Optimal-Path Precision Terrain-Following System

James E. Funk*
U.S. Air Force, Wright-Patterson Air Force Base, Ohio

A new approach to the informational processing in a terrain-following control system uses a cubic-spline curve to provide a very smooth reference path for the aircraft to follow. The spline is optimally computed to lie as close as possible to the terrain and yet to satisfy the practical constraints. Since the computed optimal path is smoother than those of other systems, the path and its derivatives can be used in a simple tracking system to provide precise control over the aircraft path. The scheme has onboard potential for advanced terrain-following systems, since splines allow consideration of fewer computational data points.

Nomenclature

a = acceleration
C = matrix in the constraint equation
c = clearance curve altitude
c_{min} = minimum-clearance distance above the terrain
D = vector indicating the constraint levels in the constraint equation
e = clearance height in excess of the minimum-clearance distance
F = frame length, horizontal range interval considered for each path optimization
fps = feet per second
G = standard sea level gravitational unit (32.17 fps = 9.807 m/sec)
H = height of the maximum expected terrain obstacle
h = height of the flight path
J = performance measure
j = jerk, time rate of change of acceleration
k = "curvature", second derivative of path height with respect to range
K = mathematical curvature of a curve ($K = k\cos^3\gamma$)
L = linear term coefficient in the performance measure
M = min-max term coefficient in the performance measure
N = number of sample points in a frame
n = index for a sample point
p = kink, third derivative of the path height with respect to range
Q = quadratic term coefficient in the performance measure
R = horizontal range variable
Γ = $[R_0, R_N]$, the range interval considered for each frame
r = normalized range on a sample interval
s = path slope
T = terrain height
V = nominal vehicle speed
Δ = sample interval for horizontal range
γ = flight-path angle
σ_T = standard deviation of the terrain sample heights
$(\ )'$ = matrix or vector transpose

Introduction

SUCCESSFUL terrain-following systems have been operating in aircraft for a number of years. As radar technology in enemy defensive systems improves, however, future terrain-following controllers must be improved to produce lower flight paths. Even though it may not be economical to install an "optimum" terrain-following control system on every aircraft or missile, it is still very desirable to know what the best possible performance of the vehicle is. The control system described here provides a practical answer to the problem of best performance, and is also a good candidate for an onboard control system in many types of advanced vehicles.

Only the terrain-following control problem (the longitudinal control of the aircraft) is discussed directly, although similar techniques can be applied to the terrain-avoidance problem (lateral control). A fundamental contention of this paper is that the "best" terrain-following controller should compute a path that satisfies two important requirements: 1) that it lies as close to all terrain points as is practical; and 2) that it can be followed extremely well by the actual aircraft. A unified processing system which satisfies these two requirements is described here. A very flexible system, it can be adapted readily to changes in mission requirements, such as aircraft speeds and maneuverability restrictions.

The terrain-following control systems that are currently on operational aircraft compute flight-path angle commands based on a "critical" point on the terrain ahead of the aircraft.[1,2]. For different systems, the methods vary for determining which point is currently the most critical. Since it is the flight-path angle, or slope, that is directly controlled, the actual vehicle path is not tightly controlled. The path is the integral of the slope with respect to range; therefore, the height error is the integral of the total slope errors, which are due to both sensor errors and control system implementation errors.

The concept of basing a series of commands on a single terrain point is simple and direct, but is rather inflexible and ineffective if the overall objective is to stay as close to all terrain points as possible. To meet this objective the path must be controlled more directly and more terrain data must be considered. Three other methods of direct path control have been proposed.[3-5] Each of these systems computes a reference path for the aircraft to follow, but each type of reference path suffers from at least one of three disadvantages: 1) the path is not smooth enough to be followed precisely by an actual aircraft; 2) the acceleration constraints may be violated in some intervals along the path; or, 3) the computation of a path segment is based on a single critical point and, thus, does not take full advantage of terrain masking over other portions of the terrain. The systems developed in Refs. 3 and 4 do not generate smooth reference paths and may have acceleration constraint violations in some regions along the path. The system in Ref. 3 processes sets of discrete points in an "optimum" manner, but the actual sense of optimality is difficult to define. The reference paths of Ref.

Received Dec. 18, 1975; presented at the AIAA Guidance and Control Conference, San Diego, Calif., Aug. 16-18 (no preprints, bound Conference Volume); revision received June 1, 1976.

Index categories: Navigation, Control, and Guidance Theory; LV/M Guidance Systems.

*Major, Avionics System Engineer, Avionics Division, Directorate of Advanced Systems Design, Deputy for Development Planning, Aeronautical Systems Division. Member AIAA.

4 may have cusps, which induce transients into the associated tracking system. The paths in Ref. 5 are composed of simple parabolic segments that clear a single critical point; this structure limits the flexibility of the path and also requires instantaneous reversals in acceleration at the segment junctions. Any abrupt transition from one type of path constraint to another can create transients in the control system and can cause fatigue problems for the aircraft structure, pilot, and engines.

The system proposed here offers a different method of optimal path processing with a readily discernable sense of optimality. Also, a smoother reference path which is a cubic spline (a series of cubic polynomial segments with continuous first and second derivatives) is computed. The spline path is effectively acceleration-limited and acceleration-rate-limited when bounds are placed on its second derivative, as will be explained fully later.

Problem Definition

The terrain-following problem that was stated very generally in the previous section will now be defined more precisely. Rather than merely determining an "acceptable" path, the problem is to determine the "best" path for the aircraft to follow. Although the definition of the best path is the subject of much debate, a reasonable criterion is the selection of the path that is "closest" to the terrain while satisfying all of the physical and operational constraints on the aircraft. A general closeness criterion that has some flexibility will be defined mathematically in terms of an excess clearance variable, e – the clearance above the terrain in excess of a specified minimum, $c_{\min}$.

$$e = h - T - c_{\min} \tag{1}$$

$$J = \int_{R_O}^{R_N} (\tfrac{1}{2}Qe^2 + Le)\mathrm{d}R + M \max_{\Gamma}\{e\} \tag{2}$$

Equating different combinations of the Q, L, and M coefficients to zero can create a quadratic, linear, or min-max performance measure that is to be minimized subject to the constraints on aircraft motion.

$$(\mathrm{d}h/\mathrm{d}R) = s \tag{3}$$

$$(\mathrm{d}s/\mathrm{d}R) = k \tag{4}$$

$$(\mathrm{d}k/\mathrm{d}R) = p \tag{5}$$

$$0 \le e \tag{6}$$

$$k_{\min} \le k \le k_{\max} \tag{7}$$

$$s_{\min} \le s \le s_{\max} \tag{8}$$

$$p_{\min} \le p \le p_{\max} \tag{9}$$

The control problem is to determine a curvature function k that minimizes J of Eq. (2) subject to the constraints of Eqs. (3 – 9). The curvature k, and the kink p are analogous to acceleration and jerk in the time domain. The "curvature" as defined by Eq. (4) is not the usual mathematical curvature variable K but is closely related to it and is more convenient to use in this problem.

$$k = (a_N/V^2)\sec^3\gamma \simeq (a_N/V^2) = K \tag{10}$$

Thus, the normal-acceleration constraints imposed on the aircraft can be approximated by constraints on the path curvature, k. The limits on k have an additional benefit of restricting the combination of normal acceleration and path slope to produce more practical paths without imposing the slope constraint limits of Eq. (8). Although the limits on acceleration rate are not usually specified for terrain following, they can be imposed to provide a smoother path. The constraints would actually be imposed on the kink, which is related to the jerk by the following approximation.

$$p \simeq j/V^3 \tag{11}$$

In the following section, this optimal control problem will be transformed into a mathematical programming problem that retains all of the essential character of the control problem, while simplifying the computation of the solution considerably.

Spline Model for the Reference Path

Inevitably, when terrain following performance is evaluated, investigators resort to sampling the trajectories and the terrain heights at various intervals. It is preferable to work in range intervals, rather than time, for then the usual assumption of constraint horizontal velocity is not required to relate the terrain locations to those of the aircraft. Thus, it is reasonable to consider discretizing the performance measure with respect to range. Furthermore, the reference path model can be discretized with respect to range, without losing its characteristic smoothness, through use of cubic splines.

The discretized equations corresponding to the continuous Eqs. (1-9) are, for $n = 1, 2, 3, \ldots, N$

$$e_n = h_n - c_n = h_n - (T_n + c_{\min}) \tag{12}$$

$$J = \sum_{n=1}^{N} [\tfrac{1}{2}Q_n e_n^2 + L_n e_n] + M \max_n\{e_n\} \tag{13}$$

$$0 \le e_n \le m \tag{14}$$

$$k_{\min} \le k_n \le k_{\max} \tag{15}$$

$$s_{\min} \le s_n \le s_{\max} \tag{16}$$

$$p_{\min} \le p_n \le p_{\max} \tag{17}$$

$$s_n = s_{n-1} + \tfrac{1}{2}\Delta(k_n + k_{n-1}) \tag{18}$$

$$h_n = h_{n-1} + s_{n-1}\Delta + (\Delta^2/6)\ (k_n + 2k_{n-1}) \tag{19}$$

$$p_n = \Delta^{-1}(k_n - k_{n-1}) \tag{20}$$

Equations (18-20) reflect the fact that the path is represented by cubic polynominals in range and the curvature is a linear spline, composed of linear segments as illustrated in Fig. 1. Note that, by recursive use of Eqs. (18) and (19), each height and slope sample value can be expressed as an affine function of the control parameters, k_n. Thus, if the coefficients Q_n are nonzero, J is a quadratic function of the k_n, but when the Q_n are all zero, the performance measure is linear. Normally the M coefficient is zero unless a min-max performance criterion is desired. The right-hand inequality of Eq. (14) involves a scalar parameter m that is selected by the minimization procedure as the smallest maximum excess clearance value that can be obtained when the min-max criterion is used. For other criteria, only the nonnegavitity portion of Eq. (14) is enforced. From Fig. 1, it can be seen that when the values of curvature are bounded by Eq. (15) at the control sample points, the curvature function is bounded over the entire interval Γ. The slope limits of Eq. (16) may not be required for many problems. Slope limits have been imposed on previous systems to eliminate maneuvers from which recoveries could not be made satisfactorily. This solution is automatically enforced by the minimization of the performance measure and the enforcement of Eqs (14) and (15). Therefore, only the

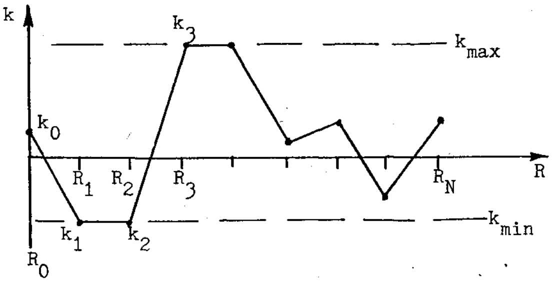

Fig. 1 Curvature spline.

special cases, such as let-downs along glide slopes and attitude restrictions on transport aircraft, would require enforcement of Eq. (16). The kink limits of Eq. (20) may not be required either, since inherent limits are imposed by the choice of the sample interval Δ and the limits on k.

$$p_{max} = \Delta^{-1}(k_{max} - k_{min}) \tag{21}$$

$$p_{min} = -p_{max} \tag{22}$$

Mathematical Programing Problem

The resulting mathematical programing problem is to determine the N elements of the $\boldsymbol{k}$ vector and the scalar m that minimize a performance measure of the matrix form

$$J = [k\ m]\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix} + [L'M]\begin{bmatrix} k \\ m \end{bmatrix} \tag{23}$$

subject to the constraints of Eqs. (14-17) which can be summarized in the form

$$C'\begin{bmatrix} k \\ m \end{bmatrix} \leq D \tag{24}$$

The m parameter and the associated maximum excess-clearance constraints are included in the problem only for the min-max criterion ($M \neq O$). The programming problem is either quadratic or linear, depending upon the selection of the coefficients in Eq. (13). The solution of either form of the mathematical programming problem is a cubic spline path that represents, in some sense, the best path for the aircraft to follow. A variety of algorithms exist for solving the programming problems on a digital computer, but a discussion of those is beyond the scope of this paper. Reference 10 contains the detailed development of the programming problem and a discussion of algorithms to solve it.

Once the N curvature values are determined by solving the programming problem, the height and slope values for the spline can be computed from Eqs. (17) and (18). The set of height and slope values can be transformed readily into continuous functions to provide the path height and all of its derivatives at any range R on the interval Γ. For $R \epsilon [R_n, R_{n+1}]$ and

$$r \triangleq \Delta^{-1}(R - R_n) \tag{25}$$

$$h(r) = h_n + r^2(3-2r)(h_{n+1} - h_n) + \Delta[(r^3 - 2r^2 + r)s_n + (r^3 - r^2)s_{n+1}] \tag{26}$$

$$s(r) = (6r/\Delta)(1-r)(h_{n+1} - h_n) + [(3r^2 - 4r - 1)s_n + (3r^2 - 2r)s_{n+1}] \tag{27}$$

$$k(r) = (6/\Delta^2)(1-2r)(h_{n+1} - h_n) + (2/\Delta)[(3r-2)s_n + (3r-1)s_{n+1} \tag{28}$$

$$p = -(12/\Delta^3)(h_{n+1} - h_n) + (6/\Delta^2)(s_{n+1}) \tag{29}$$

One practical obstacle remains in establishing a well-defined mathematical programming problem: the determination of the range intervals Γ and Δ to be used for the optimization problem.

Characteristic Maneuver

Accurate terrain data are required before the path can be optimized. To perform terrain following perfectly, the system must have sufficient data for the terrain ahead of the aircraft, but it is impractical to use all of the terrain information for the entire flight. The computational problem is too large, and all of the terrain information may not be available at the beginning of the flight. The range interval of terrain data that is used in the optimization problem is termed the frame length. A succession of optimization problems are solved as the aircraft traverses the terrain. The minimum frame length that provides good performance should be used in order to minimize the computational requirements. The frame length required depends upon the specific terrain and the

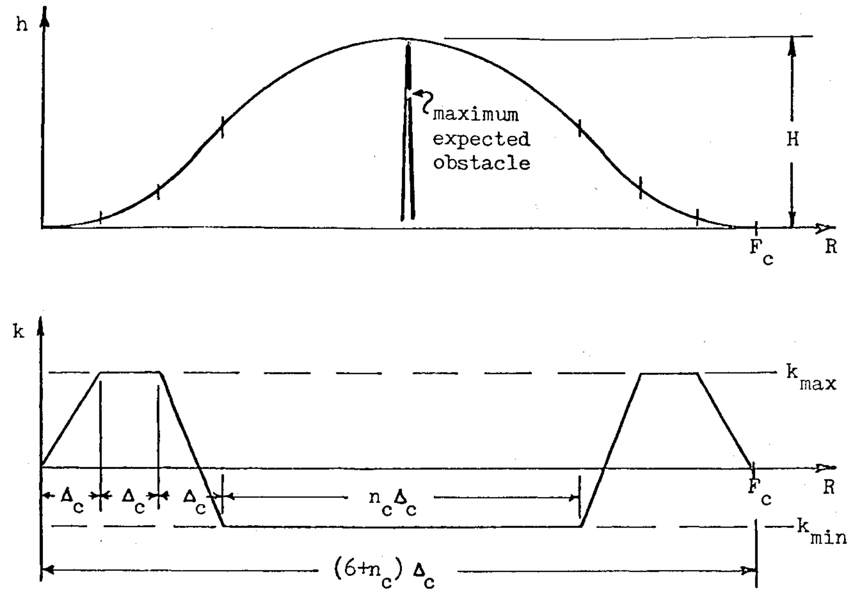

Fig. 2 Characteristic maneuver.

maneuverability of the vehicle (the allowable span of the curvatures). It is difficult to determine a minimum frame length precisely because of the unpredictability of many terrains. However, a good estimate of an adequate frame length can be computed from the simple "characteristic maneuver" defined in Fig. 2. It is a symmetric maneuver over a maximum expected terrain obstacle, of height H. The maneuver consists of a series of maximum-pullup and maximum-pushover arcs, separated by transition arcs. Each of the arcs is assumed to have a range length equal to the characteristic interval Δ_c except the central pushover arc, which has a length of $n_c\Delta_c$. Since the negative curvature frequently is restricted to a smaller magnitude than the positive curvature, the real number n_c is usually greater than one. The positive and negative areas under the curvature profile in Fig. 2 must be equal for the aircraft to resume level flight.

If the path slope restrictions are imposed upon the aircraft, these would have to be included as separate arcs in the climb and dive regions of the characteristic path of Fig. 2. For simplicity, no slope constraints are enforced here.

Both the characteristic frame length F_c and the characteristic control interval Δ_c can be determined[10] by piecewise integrations of the curvature function shown in the figure, after H, $k_{\min}$, and $k_{\max}$ are specified.

$$\Delta_c = \sqrt{\frac{24H}{28k_{\max}[1-(k_{\max}/k_{min})]+k_{\min}}} \tag{30}$$

$$F_c = \Delta_c(5-4k_{\max}/k_{\min}) \tag{31}$$

Also, n_c is a function of the curvature span.

$$n_c = -[4(k_{\max}/k_{\min})+1] \tag{32}$$

The characteristic interval represents the maximum size control interval that can be used while performing the maneuver within the frame length F_c. If a Δ is selected that is greater than Δ_c, the pullup maneuver will cause the path to rise higher than H before level flight is attained, and level flight will be attained at a range greater than $F_c/2$. To be certain that a characteristic maneuver can be performed within a frame length that is an integer multiple, n, of a control interval Δ one can choose a frame length of

$$F=(6+n)\Delta \tag{33}$$

when

$$n \geq n_c \tag{34}$$

The obstacle height H can be readily determined if complete terrain data is available for the flight. However, even when a priori data is not available, an estimate of H can be made by estimating the type of terrain that will be encountered in terms of its roughness. The terrain roughness is frequently indicated by the standard deviation of the terrain heights, T_n.

$$\sigma_T = \left[\frac{1}{N}\sum_{n=1}^{N}(T_n - T_{\text{mean}})^2\right]^{1/2} \tag{35}$$

and, thus, a reasonable estimate for H is

$$H = 3\sigma_T \tag{36}$$

Simulation of the Spline-Following Control System

A well-defined programming problem has been developed; now, the solution of that problem is considered as a means of effectively controlling the aircraft push. First, the solution to problems with various framing structures will be illustrated and discussed, then the task of following the reference paths with a simple tracking system will be considered.

Solutions of the Optimization Problem

The solutions to the optimization problems depend primarily upon the constraints imposed on each problem. The choice of a closeness criterion is not as significant as the enforcement of the constraints[10]. The programming problem solutions consist of various constraint arcs, pieced together with appropriate transition arcs. Therefore, the linear and quadratic criteria solutions are very similar, and only the quadratic performance solutions are illustrated here. The min-max paths do differ somewhat from the others, but they

Table 1 Framing structures

Framing structure number	1	2	3	4
Type of terrain		Moderately rough (σ_T = 364 ft)		Smooth (σ_T = 137 ft)
Type of ride	Hard	Hard	Soft	Hard
Expected obstacle height (1000 ft)	1.0	1.0	1.0	0.5
Characteristic frame length (1000 ft)	18.8	18.8	30.8	17.4
Frame length (1000 ft)	20	20	32	24
Frame advance distance (1000 ft)	4	4	4	12
Limits: Curvature (10^{-5} ft^{-1})	+8 −4	+8 −4	+8 −1	+6 −2
Acceleration[a] (G's)	+2 −1	+2 −1	+2 −0.25	+1.5 −0.5
Kink[b] (10^{-8} ft^{-2})	±2.66	±1.33	±1.00	±0.89
Number of control points	20	10	32	12
Complementary problem dimension[c]	60	40	64	36
Sample interval size (1000) ft:				
Characteristic	1.44	1.44	0.84	1.02
Curvature	1	2	2	2
Performance measure	1	2	2	2
Clearance constraint	1	1	1	2
Terrain data	2	2	2	2
Figure number (where structure is used)	3	3,4,&7	4	5

[a] Based on a speed of 894 fps; [b] inherent limits from curvature limits and control interval; [c] quadratic performance measure ($Q=0.1/N$, $L=M=0$).

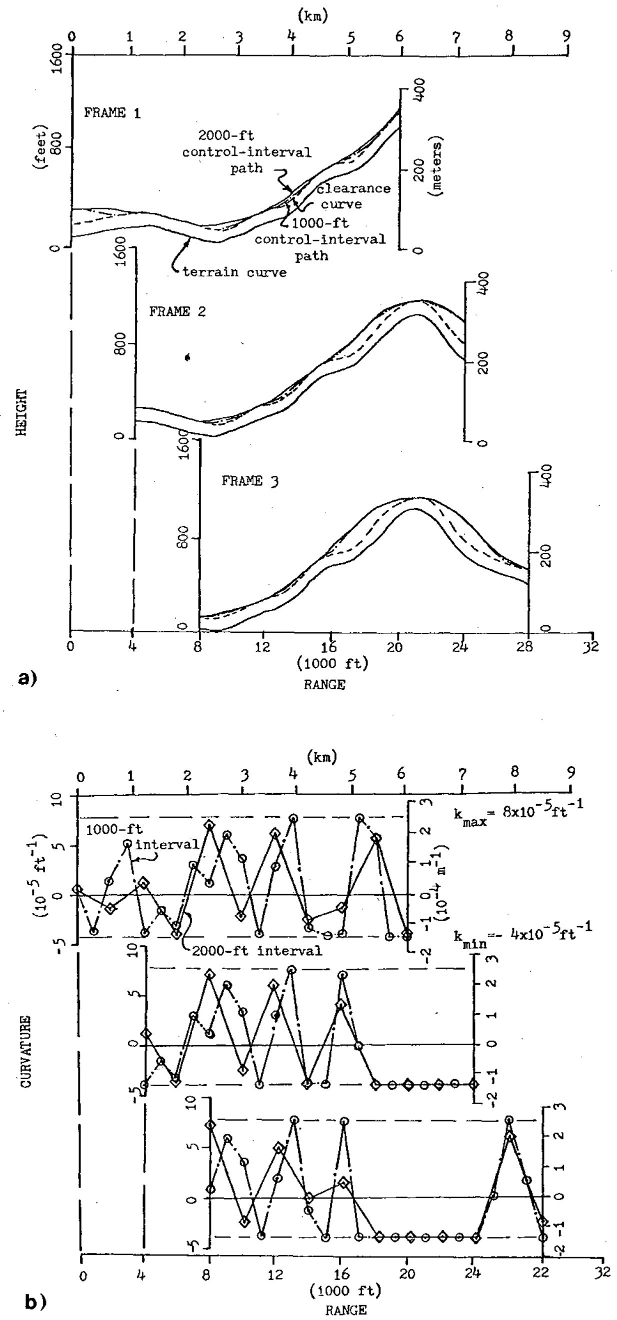

Fig. 3 a) Successive optimization solutions; b) successive optimization curvatures.

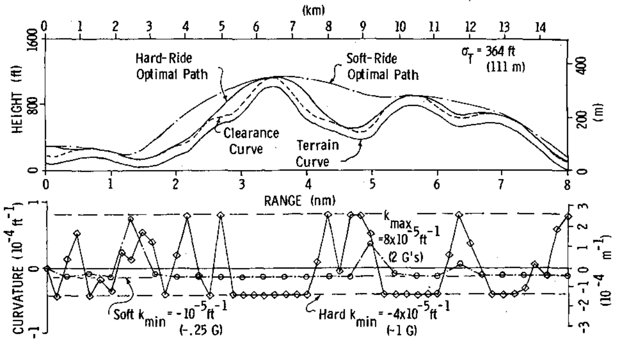

Fig. 4 Optimal paths for moderately rough terrain.

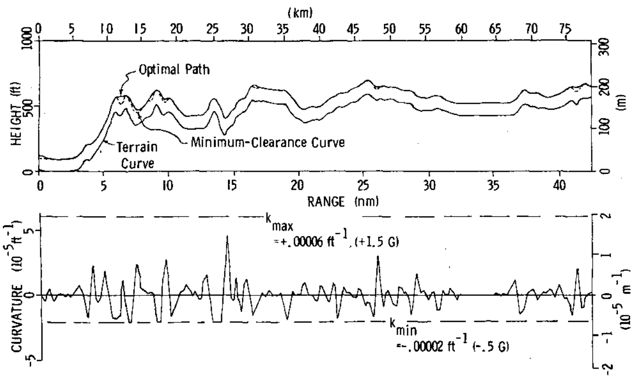

Fig. 5 Optimal path for smooth terrain.

have the disadvantage of requiring higher dimensional programming problem to obtain a path with only a slightly lower maximum excess clearance value. The particular algorithm that was used to compute the solution paths is one by Ravindran,[7] who modified the Lemke approach[9] of converting the quadratic programming problem into a linear "complementary problem" through the application of the Kuhn-Tucker conditions.[11]

The solutions of three successive optimization problems (frames) are shown in Fig. 3. The paths in Fig. 3a, and the optimal curvature profiles, in Fig. 3b, are plotted for two different control interval sizes: 1000- and 2000-ft intervals. There are only slight differences in the two types of paths, but the path using the shorter intervals does follow the terrain curve more closely. The frame length used in both cases is 20.000 ft (6096 m), compared to the characteristics frame length estimate of 18,800 ft. The detailed framing structures are listed as Structures 1 and 2 in Table 1.

Only the first 4000-ft segment of each frame, the nonoverlapping frame-advance portion, is used for guidance commands. Updated information from subsequent frames will be available for the latter portions of the path and terrain in each frame. Details of the recursive onboard computation procedure are given in Ref. 10.

Although computational times for this feasibility study are not truly representative of air-borne computation, they are somewhat indicative of the real-time system capability. More efficient programming would probably reduce these CDC 6600 computational times. A typical frame time was 3 sec. or less; however, there can be considerable variation from frame to frame. A 3-sec. frame time would allow a vehicle flying at Mach 1 to update the optimal reference path at least every 3500 ft. This path update time is independent of the cycle time for the flight control system, which would sample the reference path at a much higher rate. The main considerations in the path update time are the effects of radar shadowing and variations in sensor accuracy with range.

The composite guidance path from eight successive frames is shown in Fig. 4, along with another composite solution for a soft-ride path. The structure for the soft-ride is Structure 3 in Table 1. There is considerable difference between the hard- and soft-ride paths; the less maneuverable soft-ride path must remain high over the valleys; and it also requires a longer frame length. A 32,000-ft frame (9753.6 m) is used, compared to the characteristic length of 30,800 ft, for a 1000-ft obstacle height. That obstacle height was estimated from the terrain data that corresponds to a moderately rough terrain with $\sigma_T = 364$ ft. (The $3\sigma_T$ estimate is 1392 ft.)

Other solutions for frame lengths longer than those of Structures 2 and 3 are essentially the same as those shown in Fig. 4, while the frames appreciably shorter cause a degradation of performance[10]. Thus, there is an upper bound to the required look-ahead interval of terrain data, as well as a lower bound, for good predictive capability.

A flight over smooth terrain ($\sigma_T = 137$ ft) is illustrated in Fig. 5, where the path essentially follows the minimum-clearance curve, except for a few of the valleys which are too

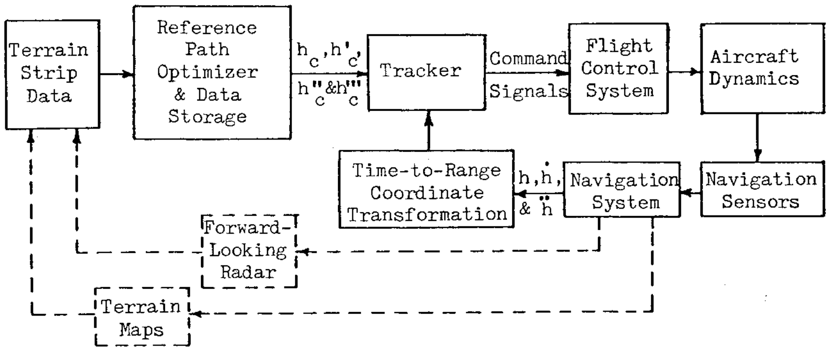

Fig. 6 Precision terrain-following system.

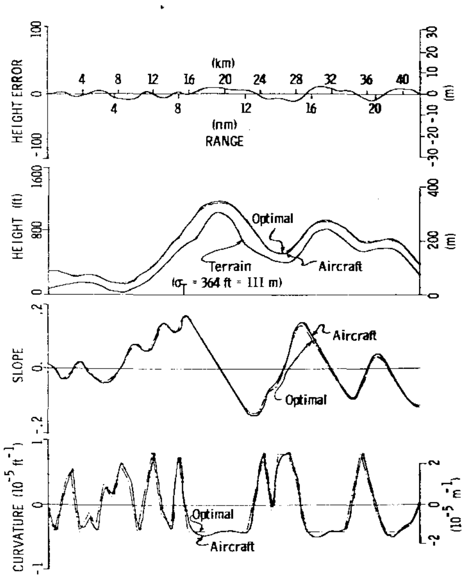

Fig. 7 Tracker performance for hard-ride with 1000-ft control interval.

steep. The corresponding framing structure is Structure 4 in Table 1. A relatively large control interval is used in this case to reduce the computational requirements by reducing the total number of sample points and, hence, the dimension of the complimentary problem. The framing structure gives adequate predictive performance.

Example solutions for a hypothetical high-speed cruise missile are discussed in Ch. 8 of Ref. 10. The results indicate that, even for this high-speed processing case, it is feasible to determine the reference flight path by onboard optimization.

Tracking the Reference Path

A variety of flight-control systems can be used with this optimized-spline processor to keep the aircraft close to the optimal path, since the spline path has two significant advantages for tracking. It was constructed with the aircraft limitations imposed, and all of the derivatives of the path are known. A feedback controller can be easily constructed to take advantage of this information.

A block diagram of the complete terrain-following control system is shown in Fig. 6. The terrain source can be forward-looking radar or stored terrain maps. The optimized path can be sequentially recomputed as new or better terrain data become available. Real time implementation is possible, since the programming problems can be solved rapidly on a digital computer. Stored sample points of the spline path and its first derivative can be used in an interpolator to generate continuous or extremely high-rate digital signals to the aircraft tracker system. The tracker compares the spline path values to those of the aircraft path, as obtained from the navigational system, and then generates control-surface commands to null the errors between the optimal and actual systems.

The ideal, or optimal, paths shown in Figs. 3-5 are indicative of effectively error-free terrain following. A nonlinear aircraft simulation is used to evaluate how well the ideal paths can be flown by an actual aircraft. Since the tracking error is of primary interest, the tracking system is provided with perfect navigational information in this simulation. The F-4C aircraft is simulated[10] and its performance is indicated in Fig. 7. The height error is the major indication of performance, but comparisons of the height, slope, and curvature of both the aircraft and optimal paths are shown. The maximum height error is approximately 12 ft, while the RMS height error is 6.5 ft.

The block diagram of the fixed-gain tracker for the simulation is shown in Fig. 8. It is possible to schedule the tracker gains according to the framing structure for a particular mission, although that was not necessary for the simulation runs of this study. The determination of gains is fairly straightforward, and the performance is not very sensitive to the gain values, since the signals to the tracker from the reference path processor are well-conditioned. Thus, the optimal spline paths can be tracked quite well by a simple tracking system.

Conclusions

The spline-path-following system can produce an aircraft path that makes maximum use of the terrain masking for maneuvers in the longitudinal plane, and a similar processor could be devised for controlling lateral path motion. Assuming good terrain data are available, the probability of impact on the terrain can be kept quite small by tightly controlling the minimum clearance above the terrain. In addition, acceleration and jerk extremes can be tightly controlled. The digital path optimizer also can be used to compute "ideal" paths as a tool in evaluating a variety of terrain-following

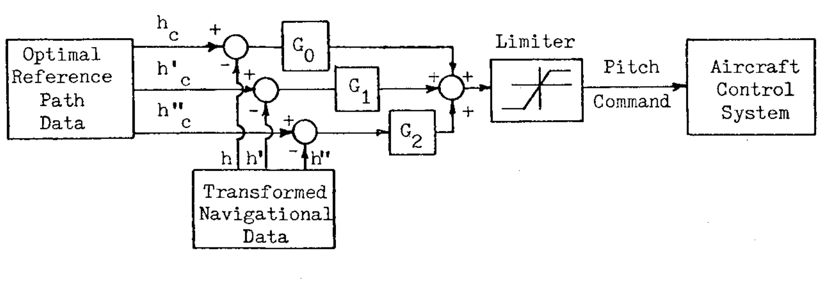

Fig. 8 Tracker system.

systems[2]. But more significantly, the computation of optimal splines is feasible for a real time airborne controller. Considering recent advances in digital computer technology, the required processing for the programing problem can be accomplished, while providing great flexibility in adapting scheme to a variety of missions and vehicles by straightforward software changes. The simple tracker system required would also be quite flexible, in that, only three gains must be adjusted to adapt to the particular vehicle dynamics. The overall performance of the system appears to be limited mainly by the accuracy of the terrain data and the navigational information that define the true state of the vehicle relative to the terrain.

References

[1] Bergmann, G. E. and DeBacker, G. E., "Terrain Following Criteria (Final Report)," Air Force Flight Dynamics Lab., Wright-Patterson AFB, Ohio, AFFDL-TR-73-135, June 1974.

[2] Jeffrie, H. L., "A Scoring Criterion for the Evaluation of Terrain Following System Performance," M. S. Thesis, St. Louis University, Mo., 1967.

[3] Greaves, J. C., "Theoretical Development of an Optimum Aircraft Control System," Ph.D. Dissertation, Rensselaer Polytechnic Institute, Troy, N. Y., Aug. 1968.

[4] Quinlivan, R. P. and Westerholt, H. H., "Investigation of Flight Control Requirements for Terrain Following (Summary)," Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio, AFFDL TR-66-65, Dec. 1964.

[5] Asseo, S. J. and Brodnick, P. J., "ADLAT VI Aircraft Control System Studies for Terrain Following/Terrain Avoidance," Air Force Avionics Laboratory, Wright-Patterson AFB, Ohio, AFAL-TR-71-134, April 1971.

[6] Ahlbery, J. H., Nilson, E. N., and Walsh, J. L., *The Theory of Splines and Their Applications,* Academic Press, New York, 1967.

[7] Ravindran, "A Computer Routine for Quadratic and Linear Programing Problems," (Algorithm 431), *Communications of ACM,* Vol. 15, Sept. 1972, pp. 818-820.

[8] Shankland, D. G., "Quadratic Programing Using Generalized Inverses," Physics Dept., Air Force Institute of Technology, Wright-Patterson AFB, AFIT-TR 75-2, Ohio, 1975.

[9] Lemke, C. E., "Bimatrix Equilibrium Points and Mathematical Programing," *Management Science,* Vol. 11, 1965, pp. 681-689.

[10] Funk, J. E., "Terrain Following Control Based on an Optimized Spline Model of Aircraft Motion," Ph.D. Dissertation, Department of Aerospace Engineering, University of Michigan, Ann Arbor, Mich., 1976.

[11] Canon, M. D., Cullum, C. D., and Polak, E., *Theory of Optimal Control and Mathematical Programing,* McGraw-Hill, New York, 1970.